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**LETTER TO THE EDITOR**

# The $SO(N)$ principal chiral field on a half-line

N J MacKay†
Department of Applied Mathematics, University of Sheffield, Sheffield, S3 7RH, UK

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**Abstract.** We investigate the integrability of the $SO(N)$ principal chiral model on a half-line, and find that mixed Dirichlet/Neumann boundary conditions (as well as pure Dirichlet or Neumann) lead to infinitely many conserved charges classically in involution. We use an anomaly-counting method to show that at least one non-trivial example survives quantization, compare our results with the proposed reflection matrices, and, based on these, make some preliminary remarks about expected boundary bound-states.

## 1. The principal chiral field

We first recall some preliminaries. A full treatment of the model on the infinite line can be found elsewhere [1]. The principal chiral model may be defined by the Lagrangian

$$\mathcal{L} = \mathrm{Tr}(\partial_\mu g^{-1} \partial^\mu g) \tag{1}$$

where the field $g(x^\mu)$ takes values in a compact Lie group $\mathcal{G}$, here chosen to be $SO(N)$. It has a global $\mathcal{G}_L \times \mathcal{G}_R$ symmetry with conserved currents

$$j(x,t)^L_\mu = \partial_\mu g g^{-1} \qquad j(x,t)^R_\mu = -g^{-1}\partial_\mu g \tag{2}$$

which take values in the Lie algebra $\boldsymbol{g}$ of $\mathcal{G}$: that is, $j = j^a t^a$ (for $j^L$ or $j^R$: henceforth we drop this superscript) where $t^a$ are the generators of $\boldsymbol{g}$. The equations of motion are

$$\partial^\mu j_\mu(x,t) = 0 \qquad \partial_\mu j_\nu - \partial_\nu j_\mu - [j_\mu, j_\nu] = 0 \tag{3}$$

which may be combined as

$$\partial_- j_+ = -\partial_+ j_- = -\tfrac{1}{2}[j_+, j_-] \tag{4}$$

in light-cone coordinates $x^\pm = \frac{1}{2}(t \pm x)$. The PCM has additional involutive discrete symmetries ('parities')

$$\pi : g \mapsto g^{-1} \Rightarrow j^L \leftrightarrow j^R \tag{5}$$

and, for $\mathcal{G} = SO(N)$,

$$\tau : g \mapsto MgM^{-1} \Rightarrow j^L \mapsto Mj^L M^{-1} \qquad j^R \mapsto Mj^R M^{-1} \tag{6}$$

where $M$ is an $O(N)$ matrix with determinant $-1$.

† E-mail address: n.mackay@sheffield.ac.uk

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## 2. Boundary conditions for the half-line

We place the model on the half-line $-\infty < x < 0$. Immediate suggestions for boundary conditions (BCs) might be the (free) Neumann condition $\partial_1 g = 0$ at $x = 0$, implying $j_1^L(0,t) = j_1^R(0,t) = 0$, or the Dirichlet condition $\partial_0 g = 0$ at $x = 0$, implying $j_0^L(0,t) = j_0^R(0,t) = 0$. Following Moriconi and de Martino [2], we generalize these to mixed conditions, with (both $L$ and $R$) $j_0(0,t) = 0$ on $M$ components of the quantum vector multiplet, and $j_1(0,t) = 0$ on the $N - M$ others. This implies

$$j_1(0,t) \in SO(M) \qquad j_0(0,t) \in SO(N-M) \tag{7}$$

which, along with $j_\mu \to 0$ as $x \to -\infty$, we shall take as our classical BCs. We have not attempted to find BCs which distinguish $L$ from $R$. Neither have we found the boundary Lagrangian corresponding to the Dirichlet conditions.

It immediately follows from (3) that

$$\begin{aligned} \partial_1 j_0(0,t) &= \partial_0 j_1(0,t) \in SO(M) \\ \partial_1 j_1(0,t) &= \partial_0 j_0(0,t) \in SO(N-M) \end{aligned}$$

and thence that *all* higher derivatives of $j_0$ and $j_1$ at $x = 0$ are in either the $SO(M)$ or the $SO(N-M)$ subgroup.

## 3. Local conserved charges on the half-line

In a recent work [1] we investigated local conserved charges on the full line, and found that the densities of the (odd-spin $s$) simple charges

$$q_s = \int_{-\infty}^{\infty} \mathrm{Tr}(j_+^{s+1}) \qquad q_{-s} = \int_{-\infty}^{\infty} \mathrm{Tr}(j_-^{s+1}) \tag{8}$$

(which arise from the conservation laws

$$\partial_- \mathrm{Tr}(j_+^{s+1}) = 0 \qquad \partial_+ \mathrm{Tr}(j_-^{s+1}) = 0 \tag{9}$$

must be generalized to certain polynomials in order to obtain a set which commutes both mutually and with the Pfaffian charge, which exists only for $N$ even, and arises from

$$\partial_-(\epsilon_{i_1 i_2 \ldots i_{N-1} i_N} (j_+)_{i_1 i_2} \ldots (j_+)_{i_{N-1} i_N}) = 0. \tag{10}$$

We first demonstrate that our BCs lead to the conservation of

$$q_{|s|} \equiv q_s + q_{-s} \tag{11}$$

on the half-line. This follows from

$$\begin{aligned} \frac{\mathrm{d}q_{|s|}}{\mathrm{d}t} &= \int_{-\infty}^{0} \mathrm{d}x\, \partial_0 \mathrm{Tr}(j_+^{s+1}) + \partial_0 \mathrm{Tr}(j_-^{s+1}) \\ &= \int_{-\infty}^{0} \mathrm{d}x\, \partial_1 \mathrm{Tr}(j_+^{s+1}) - \partial_1 \mathrm{Tr}(j_-^{s+1}) \\ &= \mathrm{Tr}(j(0,t)_+^{s+1} - j(0,t)_-^{s+1}) \end{aligned} \tag{12}$$

because $s$ is odd, the trace always is always of the form $\mathrm{Tr}(j_0(0,t)\, j_1(0,t) \ldots)$, and so vanishes. Further, from the calculation on the full line [1], we see that on the half-line

$$\begin{aligned} \{q_{|r|}, q_{|s|}\} &= (\mathrm{const})[\mathrm{Tr}(t^c j_+(0,t)^s)\, \mathrm{Tr}(t^c j_+(0,t)^r) - \mathrm{Tr}(t^c j_-(0,t)^s)\, \mathrm{Tr}(t^c j_-(0,t)^r)] \\ &= (\mathrm{const})\, \mathrm{Tr}(j_-(0,t)^{r+s} - j_+(0,t)^{r+s}) \end{aligned}$$

by completeness, and so also vanishes. The same reasoning ensures conservation and commutation of the polynomial charges. However, it does *not* imply the conservation of the Pfaffian charge, and we have found no subtler reasoning which does so. Thus, the simple charges (11) are the maximal commuting set we have found.

For the group $SO(N)$ the Pfaffian charge is the only charge which is odd under $\tau$. Further, if we had considered $SU(N)$, where conserved charges on the full line exist for $s$ both odd and even, the reasoning above would have failed to guarantee conservation of the even-$s$ charges, which are precisely those odd under $\pi$. Our suspicion is therefore that, in general, only charges which are even under all such parities are conserved on the half-line with mixed BCs, so that boundary scattering will mix parity-doublets. This is reminiscent of the situation in affine Toda field theories [3], where general BCs did not imply conservation of odd-spin charges.

Finally we consider quantum charge conservation. Here the only method available is the anomaly-counting of Goldschmidt and Witten [4], and the only non-trivial (i.e. $s > 1$) charge which this method guarantees to be conserved on the full-line, and which we have found to be classically conserved on the half-line, is that for $s = 3$. The point is that the classical conservation law is modified by quantum anomalies [4, 1] to become

$$\begin{aligned}\partial_- \operatorname{Tr}(j_+^4) + \partial_+ \operatorname{Tr}(j_-^4) = {} & c_1(\partial_+ \operatorname{Tr}(j_+\partial_+^2 j_+ + \tfrac{1}{2} j_-[j_+, \partial_+ j_+]) + \partial_- \operatorname{Tr}(j_-\partial_-^2 j_- + \tfrac{1}{2} j_+[j_-, \partial_- j_-])) \\ & + c_2(\partial_+(\operatorname{Tr}(j_- j_+) \operatorname{Tr}(j_+^2)) + \partial_-(\operatorname{Tr}(j_+ j_-) \operatorname{Tr}(j_-^2))) \\ & + c_3(\partial_+ \operatorname{Tr}(j_- j_+^3) + \partial_- \operatorname{Tr}(j_+ j_-^3)) \\ & + c_4(\partial_- \operatorname{Tr}((\partial_+ j_+)^2) + \partial_+ \operatorname{Tr}((\partial_- j_-)^2)) \\ & + c_5(\partial_- \operatorname{Tr}(j_+^2)^2 + \partial_+ \operatorname{Tr}(j_-^2)^2)\end{aligned}$$

(for some unknown $c_i$), the most general anomaly possible with the correct symmetries. On the full line it is enough for conservation that the right-hand side is a total derivative, but on the half-line we must check [2] that the coefficients of each $c_i$ individually satisfy the procedure laid out in (12). They do so, and there is thus at least one non-trivial quantum conserved charge on the half-line, which should be enough to ensure quantum integrability.

## 4. Boundary $S$-matrices and the spectrum

Recall that the bulk spectrum consists of particle multiplets with masses $m_a = 2m \sin(\frac{a\pi}{N-2})$, in representations $(V, V)$ of $\mathcal{G}_L \times \mathcal{G}_R$, where $V$ is a reducible representation of $\mathcal{G}$ whose highest component is the $a$th fundamental representation of $\mathcal{G}$. These run from $a = 1$ to $a = n - 1$ where $N = 2n + 1$, or $a = n - 2$ where $N = 2n$. There are also spinor multiplets: one in the former case, two in the latter, which form a $\tau$-doublet. In fact the particle multiplets represent a larger Yangian algebra of non-local charges, $Y_L(\boldsymbol{g}) \times Y_R(\boldsymbol{g})$, of which they are the fundamental irreducible representations.

On the half-line, if we are correct in believing that odd-parity charges are not conserved, the parity-doublets will be broken. Further, the charges $Q^a = \int j_0^a \mathrm{d}x$ which generate $\mathcal{G}_L \times \mathcal{G}_R$ were considered on the half-line by Mourad and Sasaki [5], who pointed out that

$$\frac{\mathrm{d}Q^a}{\mathrm{d}t} = j_1^a(0, t). \tag{13}$$

Those $Q^a$ corresponding to a residual $SO(N-M)_L \times SO(N-M)_R$ symmetry therefore remain conserved, as might be expected given that the BC was free for precisely those components. We have found no residue of the non-local charges and thus of the Yangian symmetry, however.

Let us finish by comparing our results briefly with the reflection matrices which, building on the work of Cherednik [6], we have constructed [7] for the scattering of the first (vector) and second (adjoint⊕singlet) (as representations of $\mathcal{G}$) multiplets of the bulk theory off the boundary. These contain a residual $SO(N-M)$ symmetry, and so might be expected to match the BCs given. The decomposition of the boundary $S$-matrices appears not to respect any residual Yangian symmetry, again as expected.

The first and second multiplet $S$-matrices take the form

$$K_1(\theta) = \tau_1(\theta)(P^- - [N-2M]P^+)$$
$$K_2(\theta) = \tau_2(\theta)(P_A^- - [N-2M-2]P_2 + [N-2M-2][N-2M+2]P_A^+) \tag{14}$$

where $\tau_1$ and $\tau_2$ are scalar prefactors [7] with no physical-strip poles, and

$$[x] = \frac{\theta + \mathrm{i}x\pi/2h}{\theta - \mathrm{i}x\pi/2h} \tag{15}$$

(with $h = 2n-2$). $P^+$ and $P^-$ are orthogonal projectors, of dimensions $M$ and $N-M$ respectively, taking the forms

$$P^+ = \left(\begin{array}{c|c} I_M & L \\ \hline 0 & 0 \end{array}\right) \qquad \text{and} \qquad P^- = \left(\begin{array}{c|c} 0 & -L \\ \hline 0 & I_{N-M} \end{array}\right) \tag{16}$$

where $L$ is an $M \times (N-M)$ matrix which cannot be determined by the boundary YBE or the crossing/unitarity condition. $P_A^+$ and $P_A^-$ project similarly onto the second-rank antisymmetric tensor, whilst $P_2$ is a mixed projector for which we have no interpretation. To describe the principal chiral model we must use

$$X_1(\theta)K_{1L}(\theta) \otimes K_{1R}(\theta)$$

where $X_1(\theta)$ is a CDD factor with a zero at $\theta = \frac{N-2M}{2h}\mathrm{i}\pi$ (and no poles on the physical strip), introduced so that the overall pole here remains simple†. (Notice also that if boundary scattering really does mix parity-doublets then some more complicated construction would be needed if we were dealing with spinor, as opposed to vector, multiplets.)

What is $L$ for our boundary conditions? Recall our comment at the end of section two, that at $x = 0$ not only $j_0$ and $j_1$ but also all their (time- *and* space-)derivatives are in either the $SO(M)$ or the $SO(N-M)$ subgroup. If the same is true of the boundary Lagrangian, then there will be no operator in the model which can link the $M$ and $N-M$ sub-multiplets, and we must have $L = 0$‡.

To understand the full pole structure, and thus the spectrum, is a longer-term project. Fusing to obtain higher projectors is a difficult calculation, and without doing so we can make no definite statements: fused scalar prefactors can easily be vitiated by cancellations in the matrix structure. However, it is simple to see that $K_a$ will have a pole factor $[N-2M+2a-2]$, and we believe that this projects onto the $SO(M)$-restriction of the $a$th antisymmetric tensor. Following the ideas of Ghoshal and Zamolodchikov [10], in which a pole at $\mathrm{i}\theta_0$ in $K_a$ leads to a boundary bound-state (BBS) of mass $m_a \cos\theta_0$, the BBS spectrum may therefore be expected, for $M < N/2$, to include states of mass

$$m'_1 = 2m \sin\left(\frac{\pi}{h}\right) \sin\left(\frac{(M-1)\pi}{h}\right)$$
$$\dots$$
$$m'_a = 2m \sin\left(\frac{a\pi}{h}\right) \sin\left(\frac{(M-a)\pi}{h}\right)$$

† A suitable $X_1$ may be obtained by setting $x = 1 + 2M - 2N$ in a minimal version (that is, with coupling-constant dependence removed) of (3.40) of Fring and Köberle [8]; we do not reproduce it here.

‡ If $L = 0$ then we might, following the $l \leftrightarrow n-l$ reciprocity noted [9] for $SU(n)$ boundary $S$-matrices broken by diagonal boundary terms to $SU(l) \times SU(n-l) \times U(1)$, expect an $M \leftrightarrow N-M$ symmetry. But this would place free and Dirichlet BCs on the same footing, with residual $SO(N-M) \times SO(M)$ symmetry, contradicting (13); further, our reflection matrices are not invariant under $M \leftrightarrow N-M$.

$$\ldots$$

$$m'_{M-1} = 2m \sin\left(\frac{(M-1)\pi}{h}\right)\sin\left(\frac{\pi}{h}\right)$$

(higher than this and the poles leave the physical strip), with the $a$th multiplet being the $a$th antisymmetric $SO(M)$ tensor. Notice that the $a$th multiplet is degenerate in mass, and has the same dimension as, the $(M-a)$th. As $a$ increases beyond $M > N/2$ the number of states falls, with a reciprocity $M \leftrightarrow h-M$, though this does not extend to the masses, whose average increases as $M$ increases, as seems natural for these BCs.

Finally we note that the scattering described here is off the boundary ground state, but reflection matrices $K_a^{[b]}(\theta)$ also exist for the scattering of the $a$th particle off the $b$th BBS. For our matrices (14) we have checked explicitly, using the bootstrap mechanism for these states, that the second BBS appears as a pole in $K_1^{[1]}$.

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